Math 307 - Differential Equations - Spring 2017

Exam 2 Solutions

Problem 1.

(a) *The characteristic equation is*

$$r^3 - 2r^2 - 5r + 6 = (r-1)(r+2)(r-3) = 0$$

so the roots are $r = 1, -2, 3$ *meaning the solution is*

$$y = c_1 e^x + c_2 e^{-2x} + c_2 e^{3x}.$$

(b) *The characteristic equation is*

$$r^4 + 8r^2 - 9 = (r^2+9)(r^2-1) = (r^2+9)(r+1)(r-1) = 0$$

so the roots are $r = -3i, 3i, -1, 1$ *meaning the solution is*

$$y = c_1 \cos 3x + c_2 \sin 3x + c_3 e^{-x} + c_4 e^x.$$

(c) *The characteristic equation is*

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$$

so the root $r = 1$ *is repeated* 2 *times meaning the solution is*

$$y = c_1 e^x + c_2 x e^x + c_2 x^2 e^x.$$

(d) *The characteristic equation is*

$$r^4 + 8r^2 + 16 = (r^2+4)^2 = 0$$

so the roots are $r = -2i, 2i$ *and they are each repeated once meaning the solution is*

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x.$$

(e) *The indicial equation is*

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = m^3 - 2m^2 - m + 2 = (m+1)(m-2)(m-1) = 0$$

so the roots are $m = -1, 1, 2$ *meaning the solution is*

$$y = c_1 x^{-1} + c_2 x + c_3 x^2.$$

(f) *The indicial equation is*

$$\begin{aligned} m(m-1)(m-2)(m-3) + 4m(m-1)(m-2) + 3m(m-1) - m + 1 &= m^4 - 2m^3 + 2m^2 - 2m + 1 \\ &= (m^2+1)(m-1)^2 = 0 \end{aligned}$$

so the roots are $m = -i, i, 1$ *where* 1 *is repeated once meaning the solution is*

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3 x + c_4 x \ln x.$$

Problem 2.

(a) *Notice that the homogeneous equation here is the same as in Problem 1c, so the only root is $r = 1$, repeated two times. Since the root is repeated, we need to use extra powers of x in the guess for the $3e^x$ part of the particular solution. The guess we should make for the particular solution is*

$$y_p = Ax^3e^x + B\cos 2x + C\sin 2x.$$

Now, we take the required derivatives

$$\begin{aligned}
y_p &= Ax^3e^x + B\cos 2x + C\sin 2x \\
y_p' &= 3Ax^2e^x + Ax^3e^x - 2B\sin 2x + 2C\cos 2x \\
y_p'' &= 6Axe^x + 3Ax^2e^x + 3Ax^2e^x + Ax^3e^x - 4B\cos 2x - 4C\sin 2x \\
&= 6Axe^x + 6Ax^2e^x + Ax^3e^x - 4B\cos 2x - 4C\sin 2x \\
y_p''' &= 6Ae^x + 6Axe^x + 12Axe^x + 6Ax^2e^x + 3Ax^2e^x + Ax^3e^x + 8B\sin 2x - 8C\cos 2x \\
&= 6Ae^x + 18Axe^x + 12Ax^2e^x + Ax^3e^x + 8B\sin 2x - 8C\cos 2x
\end{aligned}$$

and plug them into the differential equation and gather like terms

$$\begin{aligned}
y_p''' - 3y_p'' + 3y_p' - y_p &= 6Ae^x + 18Axe^x + 9Ax^2e^x + Ax^3e^x + 8B\sin 2x - 8C\cos 2x \\
&\quad -3(6Axe^x + 6Ax^2e^x + Ax^3e^x - 4B\cos 2x - 4C\sin 2x) \\
&\quad +3(3Ax^2e^x + Ax^3e^x - 2B\sin 2x + 2C\cos 2x) \\
&\quad -(Ax^3e^x + B\cos 2x + C\sin 2x) \\
&= (6A)e^x + (18A - 18A)xe^x + (9A - 18A + 9A)x^2e^x + (A - 3A + 3A - A)x^3e^x \\
&\quad +(8B + 12C - 6B - C)\sin 2x + (-8C + 12B + 6C - B)\cos 2x \\
&= 6Ae^x + (2B + 11C)\sin 2x + (11B - 2C)\cos 2x \\
&= 3e^x + 25\cos 2x
\end{aligned}$$

This gives us the system of equations

$$\begin{aligned}
6A &= 3 \\
2B + 11C &= 0 \\
11B - 2C &= 25
\end{aligned}$$

Solving this system gives us $A = \frac{1}{2}$, $B = \frac{11}{5}$, *and* $C = -\frac{2}{5}$ *so the particular solution is*

$$y_p = \frac{1}{2}x^3e^x + \frac{11}{5}\cos 2x - \frac{2}{5}\sin 2x$$

and combining with the answer from Problem 1c we get the general solution

$$y = c_1e^x + c_2xe^x + c_2x^2e^x + \frac{1}{2}x^3e^x + \frac{11}{5}\cos 2x - \frac{2}{5}\sin 2x.$$

(b) *The homogeneous part of this equation is the same as in Problem 1e, so we get that*

$$y_1 = x^{-1}, y_2 = x, y_3 = x^2.$$

For a third order equation, we set $y = u_1y_1 + u_2y_2 + u_3y_3$ and the system of equations for Variation of Parameters becomes

$$\begin{array}{ccccccc} u_1'y_1 & + & u_2'y_2 & + & u_3'y_3 & = & 0 \\ u_1'y_1' & + & u_2'y_2' & + & u_3'y_3' & = & 0 \\ u_1'y_1'' & + & u_2'y_2'' & + & u_3'y_3'' & = & \frac{f(x)}{p(x)} \end{array}$$

where $p(x)$ is the coefficient of the y''' term. Thus the system for this problem is

$$\begin{array}{ccccccccc} u_1'x^{-1} & + & u_2'x & + & u_3'x^2 & = & 0 & & (1) \\ -u_1'x^{-2} & + & u_2' & + & 2u_3'x & = & 0 & & (2) \\ 2u_1'x^{-3} & & & + & 2u_3' & = & \frac{x^2}{x^3} = x^{-1} & & (3) \end{array}$$

Taking 2 times equation (1) and subtracting x^2 times equation (3) gives

$$2u_2'x = -x$$

so

$$u_2' = -\frac{1}{2}.$$

Now, taking equation (1) and adding x times equation (2) gives

$$2u_2'x + 3u_3'x^2 = 0$$

and plugging $u_2' = -\frac{1}{2}$ in this equation gives

$$u_3' = \frac{1}{3}x^{-1}.$$

Plus this into equation (3) to get

$$u_1' = \frac{1}{6}x^2.$$

Now we integrate each of the u to get

$$\begin{aligned} u_1' = \frac{1}{6}x^2 &\implies u_1 = \frac{1}{18}x^3 + c_1 \\ u_2' = -\frac{1}{2} &\implies u_2 = -\frac{1}{2}x + c_2 \\ u_3' = \frac{1}{3}x^{-1} &\implies \frac{1}{3}\ln x + c_3 \end{aligned}$$

This gives us the general solution

$$y = \left(\frac{1}{18}x^3 + c_1\right)x^{-1} + \left(-\frac{1}{2}x + c_2\right)x + \left(\frac{1}{3}\ln x + c_3\right)x^2$$

which if we multiply out we see there's some x^2 terms that we can absorb into c_3 so the simplified answer is

$$y = c_1x^{-1} + c_2x + c_3x^2 + \frac{1}{3}x^2\ln x$$

and that a particular solution is

$$y_p = \frac{1}{3}x^2\ln x.$$

Problem 3.

(a) *Making the substitution* $v = \frac{dy}{dx}$ *we also have* $\frac{dv}{dx} = \frac{d^2y}{dx^2}$ *so that the differential equation becomes*

$$x\frac{dv}{dx} = 2(v^2 - v).$$

Separating this gives

$$\frac{1}{v^2 - v}dv = \frac{2}{x}dx.$$

Since we divided by $v^2 - v$ *we must check if the zeros of that are constant solutions. The zeros are* $v = 0$ *and* $v = 1$*, which we can easily check are actually solutions. Integrating the above equation (use partial fractions on the* v *side) gives*

$$\ln|v - 1| - \ln|v| = 2\ln|x| + C$$

and simplifying both sides gives

$$\ln\left|\frac{v-1}{v}\right| = \ln x^2 + C.$$

Exponentiate both sides to get

$$\frac{v-1}{v} = 1 - \frac{1}{v} = Cx^2$$

and solve for v *to get*

$$v = \frac{1}{1 - Cx^2}.$$

Since $v = \frac{dy}{dx}$*, we need to integrate this one more time. The way this is integrated is different if* $C > 0$ *or* $C < 0$*, so we consider the two cases separately (if* $C = 0$*, then* $v = 1$ *which we have already accounted for).*

$(C > 0)$ *Let* $C = a^2$*, then*

$$v = \frac{1}{1 - a^2x^2} = \frac{1}{(1 + ax)(1 - ax)} = \frac{1/2}{1 + ax} + \frac{1/2}{1 - ax}.$$

Integrating this gives

$$y = \int v\ dx = \frac{1}{2a}\left(\ln|1 + ax| - \ln|1 - ax|\right) + C = \frac{1}{2a}\ln\left|\frac{1 + ax}{1 - ax}\right| + C.$$

$(C < 0)$ *Let* $C = -a^2$*, then*

$$v = \frac{1}{1 + a^2x^2}$$

and integrating this gives

$$y = \int v\ dx = \frac{1}{a}\arctan ax + C.$$

Now, the constant solutions $v = 0$ and $v = 1$ give the solutions $y = C$ and $y = x + C$, so the full collection of possible solutions is

$$\begin{aligned} y &= \frac{1}{2a}\ln\left|\frac{1+ax}{1-ax}\right| + C \\ y &= \frac{1}{a}\arctan ax + C \\ y &= C \\ y &= x + C \end{aligned}$$

(b) *By the chain rule,*

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(v) = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v = v\frac{dv}{dy}$$

as desired.

(c) *Using what we found in part (b) we again make the substitution $v = \frac{dv}{dx}$ and get*

$$yv\frac{dv}{dy} = v^2 + 2v.$$

We can separate this by dividing by $y(v^2 + v)$ (!!notice that $v = 0$ and $v = -2$ are zeros of the thing we're dividing by and check that they do satisfy the differential equation!!) to get

$$\frac{1}{v+2}dv = \frac{1}{y}dy.$$

Let's briefly look at the solutions $v = 0$ and $v = -2$. These give the solutions $y = c$ and $y = -2x + c$ to the original differential equation.
Back to the differential equation: Integrate both sides

$$\ln|v + 2| = |y| + c_1.$$

Now we exponentiate both sides and end up with

$$v + 2 = c_1 y.$$

Now, since $v = \frac{dy}{dx}$ we will subtract 2 from both sides to get

$$\frac{dy}{dx} = c_1 y - 2$$

which is separable. Divide by $c_1 y - 2$ (!!we should check if this is zero, which it is at $y = \frac{c_1}{2}$ which is actually covered by the solution coming from $v = 0$ above!!) to get

$$\frac{1}{c_1 y - 2}dy = dx$$

which we integrate to get

$$\frac{1}{c_1}\ln|c_1 y - 2| = x + c_2.$$

Multiply by c_1

$$\ln|c_1 y - 2| = c_1 x + c_2$$

(we can absorb the c_1 into the c_2) and then exponentiate

$$c_1 y - 2 = e^{c_1 x + c_2} = c_2 e^{c_1 x}.$$

This gives the solution

$$y = \frac{1}{c_1}\left(c_2 e^{c_1 x} - 2\right).$$

So, collecting all of the solutions gives

$$\begin{aligned} y &= \frac{1}{c_1}\left(c_2 e^{c_1 x} - 2\right) \\ y &= c \\ y &= -2x + c \end{aligned}$$

Problem 4.

(a)

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \frac{(i\theta)^0}{0!} + \frac{(i\theta)^1}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= \frac{\theta^0}{0!} + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + \left(\frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \cdots\right) \\ &= \left(\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \\ &= \cos\theta + i\sin\theta \end{aligned}$$

(b) *Simply plug $\theta = \pi$ into Euler's formula*

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

so moving the -1 to the other side gives

$$e^{i\pi} + 1 = 0.$$

Problem 5.

(a) *Differentiate $W = y_1 y_2' - y_1' y_2$ with respect to x to get*

$$W' = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' = y_1 y_2'' - y_1'' y_2$$

Since y_1 and y_2 are solutions of $y'' + p(x)y' + q(x)y = 0$, we have

$$\begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \end{aligned}$$

which we use to replace y_1'' and y_2'' in the derivative of W. Solving for y_1'' and y_2'' gives

$$\begin{aligned} y_1'' &= -p(x)y_1' - q(x)y_1 \\ y_2'' &= -p(x)y_2' - q(x)y_2 \end{aligned}$$

now plug them in the equation for W':

$$\begin{aligned} W' &= y_1y_2'' - y_1''y_2 \\ &= y_1(-p(x)y_2' - q(x)y_2) - (-p(x)y_1' - q(x)y_1)y_2 \\ &= -p(x)y_1y_2' - q(x)y_1y_2 + p(x)y_1'y_2 + q(x)y_1y_2 \\ &= -p(x)y_1y_2' + p(x)y_1'y_2 \\ &= -p(x)(y_1y_2' - y_1'y_2) \\ &= -p(x)W \end{aligned}$$

the desired equation.

(b) *The differential equation $W' = -p(x)W$ is separable, and to separate it we divide by W. Notice that $W = 0$ is a solution of the differential equation, so now we assume that $W \neq 0$. Then after separating we get*

$$\frac{1}{W}dW = -p(x)dx.$$

We have the initial value of W at $x = x_0$ is $W(x_0)$, so incorporating the initial value into the solution we have

$$\int_{W(x_0)}^{W} \frac{1}{s}ds = \int_{x_0}^{x} -p(t)dt$$

The left-hand integral is

$$\int_{W(x_0)}^{W} \frac{1}{s}ds = (\ln|s|)|_{W(x_0)}^{W} = \ln|W| - \ln|W(x_0)| = \ln\left|\frac{W}{W(x_0)}\right|$$

and setting it equal to the right hand integral gives

$$\ln\left|\frac{W}{W(x_0)}\right| = -\int_{x_0}^{x} p(t)dt.$$

(c) *Eponentiate both sides of the last equation in part (b) to get*

$$\frac{W}{W(x_0)} = exp\left(-\int_{x_0}^{x} p(t)dt\right)$$

and multiplying $W(x_0)$ to the other side gives

$$W = W(x_0)exp\left(-\int_{x_0}^{x} p(t)dt\right)$$

which is Abel's formula!